CHANGING CONES: THEMES IN STUDENTS' REPRESENTATIONS OF A DYNAMIC SITUATION

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Researchers have identified challenges students face when modeling dynamic situations. This report discusses the results of semi-structured clinical interviews with ten prospective secondary mathematics teachers who were provided with a dynamic image of a growing and shrinking cone. We asked the students to graph the relationship between the surface area and the height of the cone. We identify four themes in the students' solution approaches and discuss the implications of these approaches. Specifically, we discuss the themes with respect to relationships between the students' solutions and their images of the growing and shrinking cone, including the extent that they leveraged this image to determine their solutions.

Keywords: Modeling; Problem Solving; Cognition

The authors of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, 2010) argued students should have the opportunity to construct and compare a multitude of relationships including those that have constant rates of change and those that have varying rates of change. They also identified modeling and reasoning quantitatively as two practices that should permeate students' mathematical experiences. These policy calls are in line with researchers who have identified students' *quantitative* and *covariational reasoning*–students conceiving situations as composed of measurable attributes that vary in tandem–as critical to numerous K-16 concepts (Ellis, 2007; Johnson, 2015a; Moore & Carlson, 2012; Thompson, 2011). These same researchers have argued that much is to be learned about how students approach dynamic situations, including the extent that these approaches are rooted in constructing structures of quantities and relationships between these quantities. Further, researchers have called for increased attention to exploring students' activities as they make sense of situations where they conceive of multiple quantities covarying (Castillo-Garsow, Johnson, & Moore, 2013; Thompson, 2011).

We discuss ten prospective secondary mathematics teachers' (heretofore referred to as students) solutions to a task in which they graphed a relationship between the surface area and height of a cone as the cone changed in size (but maintained a constant slant-angle). Using our analyses of the students' activities in semi-structured clinical interviews (Clement, 2000), we illustrate four themes in their solution approaches. With respect to these themes, we discuss how they used their images of the situation to determine a relationship between the surface area and height of a cone. Collectively, these themes provide glimpses into students' thinking as they construct relationships between quantities in what we intended to be a three-dimensional context.

Literature Review and Motivation

Saldanha and Thompson (1998) described covariation to include, "imagistic foundations for someone's ability to 'see' covariation" (p. 298). By 'see' covariation, we infer Saldanha and Thompson did not mean that covariation and quantities are independent of the mind or merely

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perceptual objects. Instead, we interpret them to refer to someone constructing and re-constructing a dynamic situation to the point that they envision it as entailing measurable attributes and an understanding or anticipation of how these attributes change in tandem. Additionally, we characterize a sophisticated image of covariation to include the capacity to 'replay' one's image of the dynamic situation while holding in mind how these attributes are changing in tandem. In this regard, how students operate and reason in what they come to understand a covariational situation is directly tied to their images of that situation.

Students' images are important for their construction of mathematical objects (Izsák, 2004; Thompson, 1994). Focusing on students' images in applied problems, Moore and Carlson (2012) examined students' activities for the purpose of determining distinguishing features of students' images with respect to the formulas and graphs that the students produced when modeling and representing situations with covarying quantities. Most relevant to the present work, the authors noted that despite the students' creations of mathematical products that an observer might deem incorrect with respect to the intended situation, these mathematical products were consistent with the *students'* images of the situations. For instance, on a task that included a box with varying base dimensions, some students envisioned a box with fixed base dimensions. As a result, these students determined a volume formula that captured a base with fixed dimensions. Based on their findings, the authors (Moore & Carlson, 2012) argued that researchers and educators should give more attention to students' images of problem contexts including determining how these images play a role in the mathematical products students construct.

Theoretical Perspective

We leverage tenets of radical constructivism by approaching knowledge as actively built up by the individual in ways that are idiosyncratic to that individual and fundamentally unknowable to another individual (von Glasersfeld, 1995). Hence, we approach quantities as personally constructed measurable attributes (Steffe, 1991). Likewise, relationships between quantities are constructed by the individual, with these relationships being influenced by the individual's understanding of each quantity and their image of the relevant situation (along with other potential influences). It follows that we do not assume that students see situations that we provide them with in the same way that we do or intend for them to do (Thompson, 2011). For instance, a student may imagine a cone growing smoothly as a video suggests, growing in discrete snapshots corresponding to adding to the cone in sections, or physically changing in some other fashion (e.g., stretching the cone as if it is made of malleable rubber).

Although we take the stance that students' knowing and thinking is fundamentally unknowable to us as researchers, we can make inferences about students' thinking based on our interpretations of their words and actions. Steffe and Thompson (2000) referred to such inferences or models as *the mathematics of students*. Our goal was to characterize students' images of a situation and the mathematical products they created based on our inferences of their activities when given a dynamic situation as described in the following sections.

Methodology

We conducted a series of three semi-structured task-based clinical interviews (Clement, 2000) with ten students (eight female, two male). The students were enrolled in a secondary mathematics teacher education program at a large university in the southeast United States. At the time of the third interview–the interview we focus on here–these students had completed their first content course in the secondary mathematics education program, as well as at least a full calculus sequence and two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with a grade of C or better. Some students had completed several additional education and mathematics courses.

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The interviews consisted of a series of tasks and problems with many tasks asking the students to construct and represent a relationship between quantities in a dynamic situation. Each interview was videotaped and these videos were digitized for analysis. Two members of the research team were present at each interview, and for each interview, the interviewers took field notes and discussed observations and insights afterwards. Upon completion of the interviews, members of the research team viewed the videos and selected instances of student activity that revealed insights into the students' thinking. The research team then met to discuss their observations and used an open (generative) and axial (convergent) approach (Strauss & Corbin, 1998) to construct tentative themes we observed across students. Upon further analysis, themes were refined by comparing and contrasting different students' activities. Through this process of constructing, refining, and re-refining, the research group reached a consensus on themes that characterized the students' activities on the cone problem.

Task Design – The Cone Problem

At the start of the interview, we presented students with a video of a growing and shrinking cone with a fixed slant angle. The height of the cone increased and decreased at a constant rate with respect to the video playback (Figure 1). We then gave the students the following prompt, "Watch the video, which illustrates a cone with a varying height. Sketch a graph of the relationship between the height of the cone and the outer surface area of the cone."

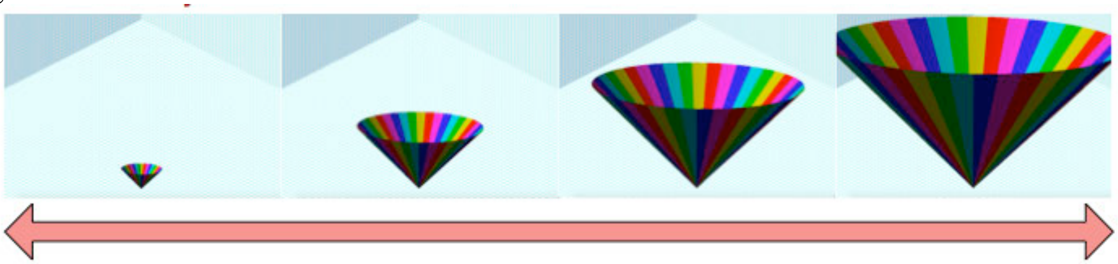

Figure 1: Dynamic Image of Cone

Saldanha and Thompson (1998) described a student's activity as he considered the relationship between two similar quantities (e.g., length)–a car's distances from two fixed points as the car traveled along a straight line. We extended this type of situation to involve the covariation of two quantities of different attributes: length and surface area. We also designed the task so that the situation lent itself to reasoning about amounts of change between the two quantities (e.g., for successive equal changes in height, the outer surface area of the cone increases and the change in the outer surface area also increases). We consider Saldanha and Thompson's (1998) task to be much more complex (imagistically) in this regard. We also note that we designed the task so that the students would not have a memorized formula readily at hand, although we hypothesized that students may attempt to construct or recall such a formula.

Results

We organize the students' solution approaches to the cone problem into four themes (Table 1). In what follows, we describe these themes including the relationships between students' images of the situation and their solution activity.

Predetermined Relationship

Students classified in the "Predetermined Relationship" theme used their initial image of the situation to dictate all further actions and conclusions. These students quickly came to a conclusion

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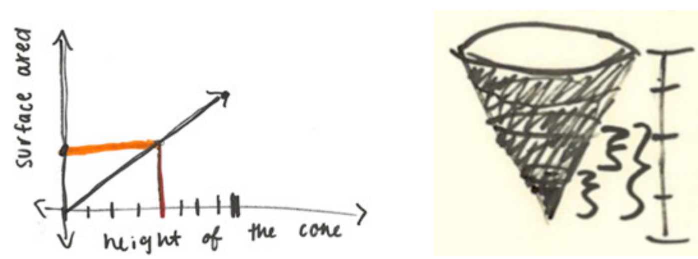
about the relationship between height and surface area based on some imagistic or physical aspect of the cone (e.g., the cone growing and shrinking in a ‘smooth’ or ‘constant’ manner). Instead of further analyzing the situation in an attempt to justify their claimed relationship, the students assimilated all

Table 1: Student Solution Approach Themes

Theme Name	Theme Description	Students
Predetermined Relationship	The student uses some imagistic aspect of the situation to reason about the relationship of the quantities. No fundamental changes of image occur from his/her initial observation.	Polly, Alice, Roz
Formula Values Determine Relationship	The student derives a formula to obtain specific numbers to determine and investigate the relationship between the quantities.	David, Kate, Terrence
Formula Structure Determines Relationship	The student derives a formula and uses properties of the formula to determine the relationship.	Angela, Audrey
Images of Covariation Determine Relationship	The student uses their image of the situation to determine the directional change and amounts of change of the two quantities under consideration.	Caroline, Trish

subsequent actions and products in terms of their initial claim. For instance, if the students determined a formula, they described the formula in terms of their initially stated relationship between the quantities; the students did not intend the formula to be for testing or verifying their relationship. Most notably, the students maintained their initial conclusions even when we directed them toward particular aspects of the situation or their activity that we thought would contradict their initial conclusions.



(a) (b)

Figure 2: Roz’s Solution to the Cone Problem

For example, after concluding the relationship was linear and drawing a graph to reflect that relationship (Figure 2a), Roz drew the 3-D cone in a way that identified (from our perspective) amounts of change of surface area for successive equal changes of cone height (Figure 2b). When asked to describe how the changes in surface area for each successive height were changing, she responded, “going off this idea that it’s a constant change in surface area,” and then described that for equal changes in height, there was an equal change in surface area. Roz maintained that the surface area changed by constant amounts, even after the interviewer repeatedly prompted her to identify and shade different sections that represented the change in surface area. This response illustrates that Roz already had a pre-determined linear relationship in mind when reasoning with her picture of the situation (Figure 2b) to identify and explain how the changes in surface area varied.

Formula Use

We identified two themes in which students constructed formulas to model surface area (but not always in relation to only the height of the cone) and used their formulas to reason about the

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relationship between height and surface area. We make two distinctions between these students' solutions based on how they used their formula to make conclusions about the relationship.

Formula Values Determine Relationship. Some students used formulas to compute numerical values. Students in this theme constructed an initial image of how the quantities were related (e.g., both the surface area and height *increase* as the cone grows), but were then perturbed by whether the quantities covary at a constant or changing rate of change. Each student then moved to determine a formula by which they could calculate paired (height, surface area) values. As no actual values were given to the students to describe the cone, the students created hypothetical values. After determining a formula, each student calculated the area for several specified values of height, with these height values increasing in equal increments. Each student then determined the relationship by comparing her or his calculated values (e.g., determining the difference between successive surface areas). We note that despite each student creating a different, technically incorrect formula, all students in this theme concluded (accurately) that the changes of surface area increased for equal changes of height.

David is one of the students who engaged in this type of reasoning. After watching the video, David conjectured that the surface area is increasing at an increasing rate with respect to height. Having difficulty using the situation or a diagram to justify his conjecture, he created a formula to compute surface area. David used the height, h, and average radius, $r_{ave,}$ (which he described as half the radius) of the cone in combination with his prior knowledge of surface area (SA) of a cylinder to derive $SA = 2\pi(r_{ave})h$. He assumed the height and radius were equal to get the final formula, $SA = 2\pi(h/2)h$. After first trying to use this formula to determine changes of surface area for arbitrary equal changes of height, he moved to using specific (numeric) height values to compute surface area values (Figure 3). He then computed differences in surface area values to conclude that the changes of surface area increase for equal changes of height.

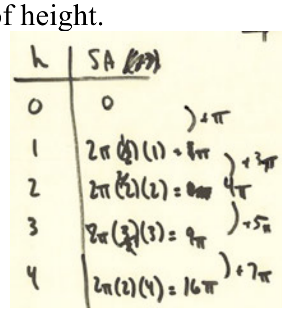

Figure 3: David's Solution to the Cone Problem

Formula Structure Determines Relationship. Whereas students grouped in the previous theme used their formulas to calculate and compare numerical values, students classified in this theme inferred how the quantities covaried based on the structure of their formulas. Specifically, the students determined a surface area formula with a multiplicative relationship between the quantities (i.e., height times height or height and radius of the base multiplied together with an assumption that the height and radius were proportionally related) and thus concluded that as height increased, surface area increased at an increasing rate.

To illustrate, after watching the video, Angela concluded that the surface area of the cone increased as the height increased. She then created a 2-D image to represent surface area by drawing a circle with a wedge removed (Figure 4a). Angela wrote the formula $A = \pi r^2$, where r represents the radius of her circle (or, equivalently, the slant height represented by x). Although she understood that the formula would not produce accurate surface area values without further information or modification, she reasoned that the formula was correct in its general structure and drew a graph (Figure 4b) that she associated with that formula. She claimed, "I know that since this is r-squared that it's going to be, the surface area is going to be a quadratic…path of a parabola" After using a

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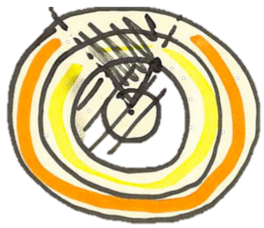
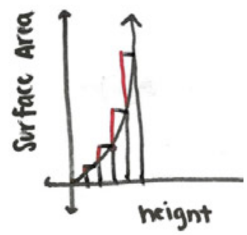
linear function to relate the radius of the circle (or, equivalently, the slant height) to the cone's height, she further justified her graph by noting that the quadratic nature of her modified formula (*A= 'some quadratic in h'*). She added, "I know what a parabola looks like and I know it's increasing at an increasing rate." Although Angela drew several diagrams to reason about an increasing surface area (see Figure 4a), she did not use these diagrams to describe *changes* in surface area for equal changes in height. Rather, Angela relied on the quadratic nature of the formula for surface area to make conclusions about the graph and relationship; she reasoned with a formula structure-covariational relationship association.





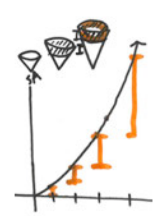
(a) (b)

Figure 4: Angela's Solution to the Cone Problem

Images of Covariation Determine Relationship

The students classified in this theme relied solely on their image of the situation to determine how the surface area and height of the cone covaried. The students maintained a 3-D image of the situation, and they were the only students to exclusively leverage a 3-D image of the situation to describe the rates of change of surface area and height of the cone. Specifically, Caroline and Trish imagined the surface area and height increasing continuously. Each student also imagined changes of surface area as successive strips for equal changes of height. The students used this image of the changes of surface area to compare successive changes in surface area.

As an example, after reading the prompt, Caroline drew three cones corresponding to equal changes in cone height (Figure 5a) and used these diagrams to determine how the quantities covaried. Caroline stated, "When you add some height, you add an extra strap around it [*re-draws 2nd cone bigger than 1st, draws strap with bottom of the strap at the height of 1st cone, seen in Figure 5b*]. Then you add some height, then you add a strap above that [*re-draws 3rd cone, bigger than 2nd with strap starting at height of top of 2nd cone seen in Figure 5b*]." She continued, "And if I were drawing these, to scale, this [*shades in strap in 3rd cone*] would have more surface area then this [*shades in strap in 2nd cone*]. So that means for equal changes in height [*marking changes of height in her diagram see in Figure 5b*], the change in surface area increases." Caroline used this reasoning to conclude that the surface area increases at an increasing rate with respect to height, and she produced a graph to reflect this relationship. Further, she identified how changes in surface area were represented on her graph (represented in orange on Figure 5c).



(a) (b) (c)

Figure 5: Caroline's Activity and Solution to the Cone Problem

Discussion

We identified themes in the students' activities that provide insight into ways these students modeled a dynamic situation. These themes are not intended to be evaluative or exhaustive; we do not claim preference of one theme over another, and there are many other possible solution approaches to this task. By comparing the students' activities across these themes, we note differences in how students' solution approaches are related to and influenced by their images of the situation. Three of the ten students (classified in the first theme) focused on a particular physical phenomenon of the situation (the constant or smooth growth of the cone), and they generalized properties of this phenomenon to the relationship between height and surface area. Five of the ten students relied on values produced from (second theme) or the attributes of (third theme) a formula to describe the relationship between the quantities. Although these five students leveraged an image of the situation that was attentive to the quantities of height and surface area, their use of this image was primarily static (e.g., using a fixed state to determine a rule). Only two of the ten students (fourth theme) used their images of the situation exclusively and continually reconstructed these images to reason covariationally.

We were surprised that only two of the ten students relied exclusively on leveraging images of the situation to represent amounts of change. Many students (e.g., Roz, Angela) engaged in activities that we believed had the potential to support them in reasoning emergently about the relationship (e.g., drawing diagrams and shading), but such activity was often assimilated in terms of previously made conclusions. This result was especially unexpected considering that in previous interview tasks several of the students classified in the first to third themes modeled relationships in dynamic situations by strictly leveraging images of the situation. One possible explanation for this result is that these students had learned the formula for the surface area of a cone in their prior school experiences. Hence, the students aimed to remember or to derive this formula rather than attempt to use images of the situation to construct the relationship.

Future Research

Several students' initial images of the situation relied on the video showing a constant change of height with respect to (implicit) time (first theme). Thus, we ponder how these students would engage in a task in which the cone's height grows at a non-constant rate or the students are able to control how the height varies. Additionally, several students persistently attempted to derive a formula for the relationship with their subsequent actions relying on their formula. Future researchers might be interested in comparing how students engage in situations that lend themselves to formulas and situations that do not. This would give insight into how students reason and rely on formulas and the consequences of such reliance, especially with respect to nuances in students' covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2015b).

Although some students maintained viable images of the quantities in the situation (from our perspective), they had difficulty reasoning about and comparing corresponding changes in the quantities. Moreover, even when we directed the students to consider how changes in the quantities might be identified in the context of the situation, some students identified what we perceived to be increasing amounts of change, yet argued that these amounts of change were equal. We envision that further investigation into students' images of change (Castillo-Garsow et al., 2013) would help explain these seemingly contradicting activities.

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